Question 2 **(5 marks)**

(a) Determine $\frac{d}{dx}\left(2xe^{2x}\right)$. (2 marks)

Solution
$\frac{d}{dx}\left(2xe^{2x}\right)=2x\left(2e^{2x}\right)+e^{2x}\left(2\right)$ $=2\left(2x+1\right)e^{2x}$
Specific behaviours
✓ uses product rule ✓ differentiates exponential term

(b) Use your answer in part (a) to determine $\int 4xe^{2x}dx$. (3 marks)

Solution
$\frac{d}{dx}\left(2xe^{2x}\right)=\left(4xe^{2x}\right)+e^{2x}\left(2\right)$ $\int\frac{d}{dx}\left(2xe^{2x}\right)dx=\int 4xe^{2x}dx+\int 2e^{2x}dx$ $2xe^{2x}=\int 4xe^{2x}dx+e^{2x}$ $\int 4xe^{2x}dx=\left(2x-1\right)e^{2x}+c$
Specific behaviours
✓ uses linearity of anti-differentiation ✓ uses fundamental theorem ✓ obtains an expression for required integral with a constant

Question 3 **(7 marks)**

Consider the function $f\left(x\right)=\dfrac{\left(x-1\right)^2}{e^x}$.

(a) Show that the first derivative is $f'\left(x\right)=\dfrac{-x^2+4x-3}{e^x}$. (2 marks)

Solution
$f'\left(x\right)=\dfrac{e^x 2\left(x-1\right)-e^x\left(x-1\right)^2}{e^{2x}}$ $=\dfrac{e^x\left(x-1\right)\left(2-x+1\right)}{e^{2x}}$ $=\dfrac{-\left(x-1\right)\left(x-3\right)}{e^x}$ $=\dfrac{-x^2+4x-3}{e^x}$
Specific behaviours
✓ uses quotient rule ✓ simplifies expression

(b) Use your result from part (a) to explain why there are stationary points at $x=1$ an $x=3$. (2 marks)

Solution
$f'\left(x\right)=\dfrac{-\left(x-1\right)\left(x-3\right)}{e^x}$ $f'\left(1\right)=0=f'\left(3\right)$
Specific behaviours
✓ identifies stationary points as $f'\left(x\right)=0$ ✓ shows that this is true for $x=1,3$

It can be shown that the second derivative is $f''(x) = \dfrac{x^2 - 6x + 7}{e^x}$.

(c) Use the second derivative to describe the type of stationary points at $x = 1$ and $x = 3$.
(3 marks)

Solution
$f''(x) = \dfrac{x^2 - 6x + 7}{e^x}$ $f''(1) = \dfrac{2}{e}$ $f''(3) = \dfrac{-2}{e^3}$ when $x = 1$ $f'' > 0$ hence local minimum when $x = 3$ $f'' < 0$ hence local maximum
Specific behaviours
✓ evaluates second derivatives for $x = 1$ and $x = 3$ ✓ uses sign to determine nature ✓ states nature for each stationary point

Question 6 **(4 marks)**

The graphs $y = 6 - 2e^{x-4}$ and $y = -\frac{1}{4}x + 5$ intersect at $x = 4$ for $x \geq 0$.

Determine the exact area between $y = 6 - 2e^{x-4}$, $y = -\frac{1}{4}x + 5$ and the y axis for $x \geq 0$.

Solution
$A = \int_0^4 \left(6 - 2e^{x-4} - \left[-\frac{1}{4}x + 5\right]\right) dx$ $= \int_0^4 \left(-2e^{x-4} + \frac{1}{4}x + 1\right) dx$ $= \left[-2e^{x-4} + \frac{x^2}{8} + x\right]_0^4$ $= (-2 + 2 + 4) - (-2e^{-4})$ $= 2\left(2 + \frac{1}{e^4}\right)$
Specific behaviours
✓ sets up an appropriate integral for area ✓ uses correct limits ✓ anti-differentiates correctly ✓ calculates area

Question 15 **(10 marks)**

The volume $V(h)$ in cubic metres of liquid in a large vessel depends on the height h (metres) of the liquid in the vessel and is given by

$$V(h) = \int_0^h e^{\left(-\frac{x^2}{100}\right)} dx,\ 0 \le h \le 15.$$

(a) Determine $\frac{dV}{dh}$ when the height is 0.5 m. (2 marks)

Solution
$V'(h) = e^{\left(-\frac{h^2}{100}\right)}$ So $V'(0.5) = e^{-0.0025} = 0.9975 m^3/m$
Specific behaviours
✓ uses FTC ✓ obtains correct value for the rate of change

(b) What is the meaning of your answer to Part (a)? (1 mark)

Solution
It means the rate of change of the volume with respect to height when the height has reached 0.5 metres.
Specific behaviours
✓ states meaning

(c) The height h of the liquid depends on time t (seconds) as follows:

$$h(t) = 3t^2 - t + 4, t \ge 0.$$

(i) Determine $\frac{dh}{dt}$ when the height is 6 m. (2 marks)

Solution
Now $h(t) = 3t^2 - t + 4 = 6 \Longrightarrow 3t^2 - t - 2 = 0 \Longrightarrow (3t+2)(t-1) = 0$ So $t = 1\ s$. Then $\frac{dh}{dt} = 6t - 1$ $\left.\frac{dh}{dt}\right\rvert_{t=1} = 6(1) - 1 = 5$ m/s
Specific behaviours
✓ differentiates h wrt t correctly ✓ state equation for time and substitutes values correctly

(ii) Use the chain rule to determine $\frac{dV}{dt}$ when the height is 6 m. (2 marks)

Solution
$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$ $=e^{-\frac{6^2}{100}}\times 5$ ≈ 3.488 m^3/s
Specific behaviours
✓ demonstrates use of the chain rule ✓ substitutes values correctly to determine rate of change

(iii) Given the volume of the liquid at 2 seconds is 8.439 m^3, use the incremental formula to estimate the volume 0.1 second later. (3 marks)

Solution
$h(2)=3(2)^2-2+4=14$ $\frac{\delta V}{\delta t}\approx\frac{dV}{dt}$ $\delta V\approx e^{-\frac{14^2}{100}}\times 11\times\delta t$ $\approx 1.54944\times 0.1$ ≈ 0.155 $V\left(t=2.1\right)\approx 8.439+0.155$ ≈ 8.594 m^3
Specific behaviours
✓ determines $h(2)$ ✓ uses incremental formula to approx. dV ✓ estimates new V

Question 16 **(8 marks)**

A group of biologists has decided that colonies of a native Australian animal are in danger if their populations are less than 1000. One such colony had a population of 2300 at the start of 2011. The population was growing continuously such that $P = P_0 e^{0.065t}$ where P is the number of animals in the colony t years after the start of 2011.

(a) Determine, to the nearest 10 animals, the population of the colony at the start of 2014. (2 marks)

Solution
$P(t) = 2300e^{0.065t}$ $P(3) = 2300e^{0.065(3)}$ $= 2795.2$ ≈ 2800
Specific behaviours
✓ determines equation for P ✓ determines population correct to nearest 10

(b) Determine the rate of change of the colony's population when $t = 2.5$ years. (2 marks)

Solution
$\frac{dP}{dt} = 0.065 \times 2300e^{0.065t}$ $\left.\frac{dP}{dt}\right\rvert_{t=2.5} = 175.879$ ≈ 176 animals/year
Specific behaviours
✓ determines derivative ✓ determines rate at 2.5 years

(c) At the beginning of 2017, a disease caused the colony's population to decrease continuously at the rate of 8.25% of the population per year. If this rate continues, when will the colony become 'in danger'? Give your answer to the nearest month. (4 marks)

Solution
$P(6) = 2300e^{0.065(6)}$ ≈ 3397 Population from 2017: $P(t) = 3397e^{-0.0825t}$ $1000 = 3397e^{-0.0825t}$ $t = 14.8$ October 2031
Specific behaviours
✓ determines population at the beginning of 2017 ✓ states new population equation ✓ solves for t ✓ determines correct month and year

Question 9 **(8 marks)**

The concentration, C, of a drug in the blood of a patient t hours after the initial dose can be modelled by the equation below.

$$C = 4e^{-0.05t} \text{ mg/L}$$

Patients requiring this drug are said to be in crisis if the concentration of the drug in their blood falls below 2.5 mg/L.

A patient is given a dose of the drug at 9 am.

(a) What was the concentration in the patient's blood immediately following the initial dose? (1 mark)

Solution
Initial dose when $t = 0$ $C(0) = 4$ mg/L
Specific behaviours
✓ determines concentration, including the unit

(b) What is the concentration of the drug in the patient's blood at 11.30 am? (2 marks)

Solution
$C = 4e^{-0.05(2.5)}$ $C = 3.53 \text{ mg/L}$
Specific behaviours
✓ substitutes $t = 2.5$ ✓ calculates concentration

(c) Find the rate of change of C at 1 pm. (2 marks)

Solution
$\dfrac{dC}{dt} = -0.2e^{-0.05t}$ $\left.\dfrac{dC}{dt}\right\rvert_{t=4} = -0.164 \quad \text{mg/L/hour}$
Specific behaviours
✓ finds derivative of C wrt t ✓ calculates rate of change when $t = 4$

(d) What is the latest time the patient can receive another dose of the drug if they are to avoid being in crisis? (3 marks)

Solution
$2.5 = 4e^{-0.05t}$ $t = 9.4 \text{ hours}$ Latest time = 6:24 pm (6:25 too late)
Specific behaviours
✓ substitutes $C = 2.5$ ✓ solves for t ✓ states latest time

Question 14 **(5 marks)**

(a) The table below examines the values of $\frac{a^h - 1}{h}$ for various values of a as h approaches zero. Complete the table, rounding your values to five decimal places. (2 marks)

h	$a = 2.60$	$a = 2.70$	$a = 2.72$	$a = 2.80$
0.1	1.00265	**1.04425**	1.05241	1.08449
0.001	0.95597	0.99375	**1.00113**	**1.03015**
0.00001	0.95552	**0.99326**	**1.00064**	1.02962

Solution
See table
Specific behaviours
✓ correctly completes three table values ✓ correctly completes all entries and rounds to 5dp

It can be shown that $\frac{d}{dx}(a^x) = a^x \lim_{h \to 0}\left(\frac{a^h - 1}{h}\right)$.

(b) What is the exact value of a for which $\frac{d}{dx}(a^x) = a^x$? Explain how the above definition and the table in part (a) support your answer. (3 marks)

Solution
$a = e \approx 2.71828$ When $a = e$ the table shows that the value of $\lim_{h \to 0}\left(\frac{a^h - 1}{h}\right)$ is 1. It follows then from the definition that $\frac{d}{dx}(e^x) = e^x \times 1 = e^x$.
Specific behaviours
✓ states $a = e$ or 2.71828 ✓ explains table result ✓ explains significance of table result for part (b)